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# Examining Functions under Quasi-Stereographic Projections

**Research Article**

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**Abstract:** We consider a generalization of Stereographic Projections, to be called "Quasi-Stereographic" Projections. These refer to projections from compact Riemann surfaces to the a plane that intersects the surface. We will observe behavior of functions, such as polynomials, when this projection is applied. This includes approximating integrals of functions via projecting the function onto the Riemann Sphere. We will also briefly consider when the surface is not compactable and mention briefly about higher dimensional Quasi-Stereographic Projections.

**Keywords:** Integration • Projections • Analysis • Differential Geometry • Mathematics



## 1. Introduction

The stereographic projection is defined as the map $P : \mathbb{R}^2 \to \mathbb{S}^2$, where $\mathbb{R}^2$ is two dimensional Euclidean space and $\mathbb{S}^2$ is defined as the unit sphere. Stereographic projections have been studied as a valid bijection between the Cartesian plane and the unit sphere by many mathematicians, including Bernhard Riemann, who eventually extended the interpretation of the stereographic projection to what it is today [1–3].

Geometrically, this projection can be described in the following way: let the unit sphere $\mathbb{S}^2$ sit such that its great circle $C \in \mathbb{R}^2$. Start with the north pole, or the top vertex of the sphere. If you draw a line from the north pole such that it intersects the sphere and the Cartesian plane at one point [4].

Our main objective is to study polynomials and other functions when mapped to compact Riemann surfaces. In particular, we will examine the properties of integrals when mapped to the 2-sphere. Using the original stereographic projections, we will show that it is possible to approximate improper integrals (with bounds from $(0, \infty)$ accurately. In addition to our analysis of the stereographic projection, we will develop a suitable projection to surfaces such as tori and other compact Riemann surfaces. A visualization of this projection can be seen in Figure 1.

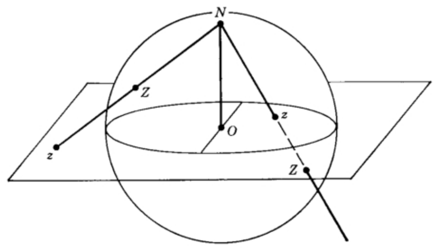


**Figure 1.** **Stereographic Projection (Retrieved from: imgur.com).**

## 2. Stereographic Projections

**Lemma 2.1.**
*(i) If the stereographic projection of a point $(\alpha, \beta, \gamma)$ to $(X, Y)$ is defined, then*

$$(X, Y) = \left(\frac{\alpha}{1-\gamma}, \frac{\beta}{1-\gamma}\right)$$

*(ii) If $(X, Y) \in \mathbb{R}^2$ is an inverse stereographic projection that maps the point to $(\alpha, \beta, \gamma) \in \mathbb{S}^2$, which is*

$$(\alpha, \beta, \gamma) = \left(\frac{2X}{1+X^2+Y^2}, \frac{2Y}{1+X^2+Y^2}, \frac{X^2+Y^2}{1+X^2+Y^2}\right)$$

*Proof.* $(i)$ Let $S(P)$ be the stereographic projection of point $P$. Let $L(t)$ represent the line that intersects the North Pole $(\alpha, \beta, \gamma)$ and the Cartesian plane. Define $L(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Because $L(t)$ will intersect the plane at the coordinate $(\alpha, \beta, 0)$ (for some $\alpha, \beta$), we can define $L(1) = \begin{bmatrix} \alpha \\ \beta \\ \gamma - 1 \end{bmatrix}$. Therefore, $L(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta \\ \gamma - 1 \end{bmatrix} t$. The intersection between $L(t)$ and the plane gives us that $S(\alpha, \beta, \gamma) = (\frac{\alpha}{1-\gamma}, \frac{\beta}{1-\gamma})$ □

$(ii)$ The second case is analogous to the first. Given $(x, y) \in \mathbb{C} \cong \mathbb{R}^2$, we can define the line from $(x, y, 0)$ to north pole as $L(t) = (0, 0, 1) + (x, y - 1)t$ such that $L(0) = (0, 0, 1)$ and $L(1) = (x, y, 0)$. This will intersect the sphere when the relation $x^2 + y^2 + z^2 = 1$ is satisfied, so we are enabled to substitute in our parameters for $x, y$, and $z$ into that equation in terms of $t$. We obtain, after computation,

$$(x^2 + y^2 + 1)t^2 - t + \frac{1}{4} = 0.$$

This only has roots when when $t = \frac{2}{x^2+y^2+1}$. Plugging this back into our parameters, we obtain our desired result. □

### Corollary 2.1.

*Any function will converge at the point $(0,0,1)$ when inversely stereographically projected, unless it is logarithmic.*

*Proof.* The lemma is proven because of the following statements:

$$\lim_{X\to\infty} \frac{2X}{1+X^2+(\sum_{i=0}^{n} a_i X^i)^2} = 0$$

$$\lim_{X\to\infty} \frac{2(\sum_{i=0}^{n} a_i X^i)}{1+X^2+(\sum_{i=0}^{n} a_i X^i)^2} = 0$$

$$\lim_{X\to\infty} \frac{X^2+(\sum_{i=0}^{n} a_i X^i)^2}{1+X^2+(\sum_{i=0}^{n} a_i X^i)^2} = 1$$

This result holds for all real functions, with the exception of logarithmics. □

Given the above lemmas, we define the north pole $N(0,0,1)$ to be the point that projects towards infinity. This projection of a point at infinity drove me to consider projecting functions onto the Riemann sphere to integrate them if their integrals were convergent. I projected points on the function's graph on the Cartesian plane onto the Riemann sphere and interpolated the curve with a $5^{\text{th}}$ order polynomial. I then used line integration on the curve formed above the points to obtain an approximation for the integral on the Cartesian plane. The integral is overestimating the area by a factor of approximately 0.5 when projecting onto the sphere (some of the beginning points project to the equator of the Riemann sphere, which increases the value of the integral by two-fold). Therefore, we have the following proposition:

### Proposition 2.1.

*Any analytic function $f$'s integral with bounds from a constant to infinity and is convergent can be approximated the following way:*

$$\left(\int_R S_f \; \boldsymbol{dt}\right) \cdot 0.5,$$

*m where $S_f$ is the function $f$'s inversely stereographically projected interpolated function.*

This proposition has been thoroughly tested previously with several examples. First, I tested the Gaussian integral, or the standard distribution function $f(x) = e^{-x^2}$ which has a well known integral of $\frac{\sqrt{\pi}}{2} \approx 0.88623$, where $x$ is integrated from 0 to $\infty$. A graph for the stereographically projected integral is provided in Fig. 2. The full calculation is shown below, using the proposition:

$$0.5 * \cdot \int_0^1 1.108 - 0.6114x^3 - 1.1358(0.5 - 1.9007x - 4.002x^2 + 28.3726x^3 - 44.6794x^4 + 22.5096x^5)^3 \; \mathbf{d}x$$

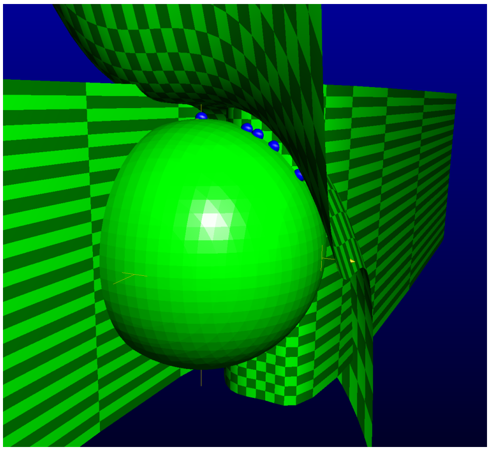

**Figure 2.** **Line Integration.**

$$\cdot \int_0^1 \sqrt{0.9007 - 8.044x + 85.1178x^2 - 178.718x^3 + 112.548x^4}\, \mathbf{d}x$$

$$\approx 0.821275$$

Using *Grapher* on MacOS, an image of this integration is given in Figure 2

Using the same procedure as above, the Dirichlet Integral's result is calculated to be $\approx 1.53256$, while the actual integral evaluates to be $\frac{\pi}{2} \approx 1.5708$ (The Dirichlet Integral is $\int_0^\infty \frac{\sin x}{x}\ \mathbf{d}x$).

Towards the future, we look to continue to improve our definition for $S_f$, or the stereographically projected polynomial. Rather than claiming it as an interpolated function, we can proceed to find a stronger definition for it using spherical coordinates.

Note that *Proposition 2.1* holds for analytic functions, it will be an alternative approximation method for improper integrals. Analytic Functions are functions that can be locally given a convergent power series.

# 3. Quasi-Stereographic Projections

## 3.1. Torus, $Q_1$

The first Quasi-Stereographic projection involves the projection from the torus to the Cartesian plane. Let this projection be known as the "$Q_1$" stereographic projection. We define it using the fundamental parallelogram, which is the following:

**Definition 3.1 (Fundamental Parallelogram).**
Fix two complex numbers $z_0, z_1 \in \mathbb{C}\ /\ \{0\}$, with $z_1/z_0 \notin \mathbb{R}$. We will let $G$ be the set of points such that

$$G = \{nT_{z_0} + mT_{z_1} \mid n, m \in \mathbb{Z}\},$$

where $T_{z_0}(z) = z + z_0$. Since we then have the isomorphism $G \cong Z \times Z$ defined by

$$(nT_{z_0} + mT_{z_1}) \to (n, m).$$

Furthermore, the quotient manifold $\mathbb{C}/G$ is defined as the torus, or $\mathbb{T}^2 = S^1 \times S^1$. This atlas, or map is to the fundamental parallelogram [2].

The $Q_1$-stereographic projection is defined as the projection from the Riemann Sphere to the fundamental parallelogram of the unit torus (we define unit torus as $(T = (z, w) \in C_2 : |z| = |w| = 1)$. Since we are considering the unit torus, its fundamental parallelogram would just be a unit square.

In short, we are scaling down from the function on the Cartesian plane to the fundamental parallelogram of the torus (Fig. 3).

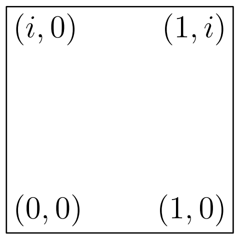


**Figure 3.** **Fundamental Parallelogram of Unit Torus**

The simplest analogue of the $Q_1$ quasi-stereographic projection is the projection from the unit torus' fundamental parallelogram to the Cartesian plane. Since this parallelogram is just a square in this special case, we can just assume that $(0, 0)$ maps to itself, $(i, 0)$ maps to our projection point at infinity on the y-axis, $(0, 1)$ as our projection point at infinity on the x-axis.

We utilized a $Q_1$ stereographic projection on an analog of the Mobius transformation. A Mobius transformation is referred to as the stereographic projection from a sphere to a plane, and is then stereographically projected to another sphere. This is a type of linear transformation in the following way:

$$z \to \frac{az + b}{cz + d},$$

where $ad - bc \neq 0$. In this study, we attempted a Mobius transformation from the first Riemann sphere to a unit torus instead, we will consider a Mobius transformation from the first Riemann sphere to the unit torus. To describe it in a different way, the sphere is "wrapping" $\mathbb{R}^2$ onto its surface.

Refer to Fig. 4 for a visualization of this transformation. We utilized our topographical definition of the torus (fundamental parallelogram) and perform our transformation, for simplicity, scaling down the sphere to a parallelogram. Since it can be thought of as a 1 by 1 square of a grid, It is a scaling, so we can project from the Riemann sphere a scaled down version of the Cartesian plane (such that the real and imaginary unit of each

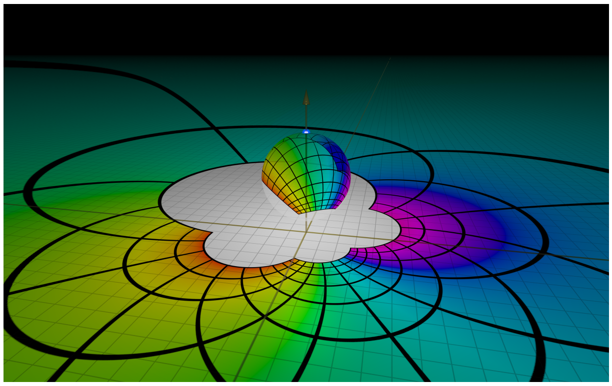

Figure 4. **Mobius Transformation (Retrieved from: math.umn.edu).**

number is between 0 and 1). We are simply representing our point at infinity in the $x$-direction as $(i, 0)$. This projection will simply be the graph of the function on the Cartesian plane, but every point at infinity will contain an endpoint. Functions on $\mathbb{R}^2$ that have a limit at infinity will have an endpoint at $(1, i)$. Otherwise, the function will converge at its limit. If the function has a limit of 0, then the function will converge at the point $(1, 0)$. To restate, the limit at infinity will be a point on the transformed function. Let us take the function on the Cartesian plane $f(X) = Y$. The graph of the transformation is shown below in Fig. 5.

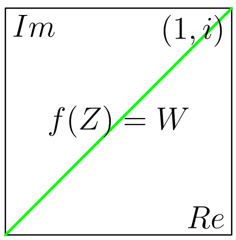


Figure 5. **Fundamental Parallelogram of Unit Torus**

To analyze this graph on the torus, we can topologically morph the parallelogram back into a torus. Folding this back into a cylinder would yield a line being curved along from the base to the top, making a full revolution about the cylinder. When the top and bottom are connected, it will be a stretched out contour that curves around the torus four times symmetrically. This finding is significant because it shows a way of projecting a function onto a torus.

Using our alternative definition of a Mobius transformation, we are wrapping the plane into a torus. Thus, as we take other tori with wider fundamental parallelograms, this wrapping will be more stretched out. For tori with equal parameters, the wrapping should be symmetric. Logically, the formation of the Quasi-Stereographic projection is understandable.

### 3.2. Non-Compact Riemann Surfaces, $Q_n$

Non-Compact Riemann surfaces are more tough to deal with, for these are not closed and the surface itself will be part of the Complex plane. But, Quasi-Stereographic projections for non-compact Riemann surfaces are nonexistent. This is because of the very fact that we will need to take the inside of the sphere to be part of the complex plane. Higher dimensional surface as well are tougher to deal with as well, for they are harder to visualize and to create the projection for.

## 4. Conclusion

In this paper, we have discussed our approximation for integrals of convergent analytic functions, which involved taking the line integral of our inversely stereographically projected on the Riemann sphere. Next, we developed a new $Q_1$ stereographic projection that projects from $\mathbb{R}^2$ to the unit torus. To do so, we consider the torus' fundamental parallelogram. Although we do not cover other tori with not as symmetric parallelograms, the projection can be performed the same way (except the coordinates of the parallelogram and the elasticity of the graph would be different).

A potential application of the infinite integral approximation is in physics and statistics, with wave density functions. We see that the wave density function's ($\Psi$) integral from $\int_{-\infty}^{\infty} \Psi * \Psi \; dx = 1$. This is important when we want to approximate the probability of an electron being located somewhere in its electron cloud. In the future, we hope to utilize the $Q_1$-stereographic projection to prove a theorem in mathematics or any parallel field. We will continue to extend and to crystallize our understanding of our projection at the same time.

## Conflict of Interest

Authors of this article declare that they have no conflict of interest.

## Human Studies/Informed Consent

No human studies were carried out by the authors for this article.

## Animal Studies

No animal studies were carried out by the authors for this article.

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